Finding the Seed of Uniform Attachment Trees

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Introduction

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This problem was popularized by Shah-Zamah [6].

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- T_i is obtained by joining vertex i to a vertex of T_{i-1} chosen uniformly at random, independently of the past.

Influence of the seed

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Curien, Duquesne, Kortchemski and Manolescu: YES. [4]

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- or a small a set $H_2(T_n, \epsilon)$ such that

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Given $T_n(P_\ell)$, $n >> 1$ we can find a set
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Theorem 2: $S_\ell = E_\ell$

For $\ell \geq \max\left\{C, \frac{8}{\gamma}\right\} \log \frac{1}{\epsilon}$ we have the following:
Given $T_n(E_\ell)$, $n >> 1$ we can find a set
$H_n = H_n(T_n, \varepsilon) \subset \{1, \ldots, n\}$ with $|H_n| \leq (1+\gamma)\ell$ such that

$$\mathbb{P}\left\{H_n \supset E_\ell\right\} \geq 1 - \epsilon \ .$$

Theorem 3: $S_\ell = T_\ell$

There exist c_1 and c_2 such that the following holds. Let
$\ell \geq c_1 \log^2 \frac{1}{\epsilon}$.
Given $T_n(T_\ell)$, $n >> 1$ we can find a set
$H_n = H_n(T_n, \varepsilon) \subset \{1, \ldots, n\}$ with $|H_n| \geq \ell/[c_2 \log(\ell/\epsilon)]$ such that

$$\mathbb{P}\{H_n \subset T_\ell\} \geq 1 - \epsilon\ .$$

Theorem 4

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Let $\epsilon \in (0, e^{-e^2})$*. Suppose that* T_n *is a uniform attachment tree with seed* $S_\ell = P_\ell$ *or* $S_\ell = E_\ell$ *for* $\ell \leq \frac{\log(1/\epsilon)}{\log\log(1/\epsilon)}$*. Then, for all* $n \geq 2\ell$*, any seed-finding algorithm that outputs a vertex set* H_n *of size* ℓ *has*

$$\mathbb{P}\left\{|H_n \cap S_\ell| \leq \frac{\ell}{2}\right\} \geq \epsilon \ .$$

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Let us define what means be more central.

Rooted tree and Induced subtree

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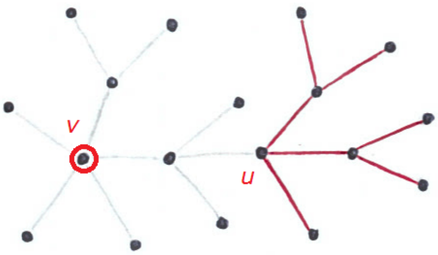

Centrality: Definition

Given a tree T, the *anti-centrality* of a vertex $v \in V(T)$ is defined by

$$\psi(v) = \max_{u \in N(v)} |(T, v)_{u\downarrow}| \ .$$

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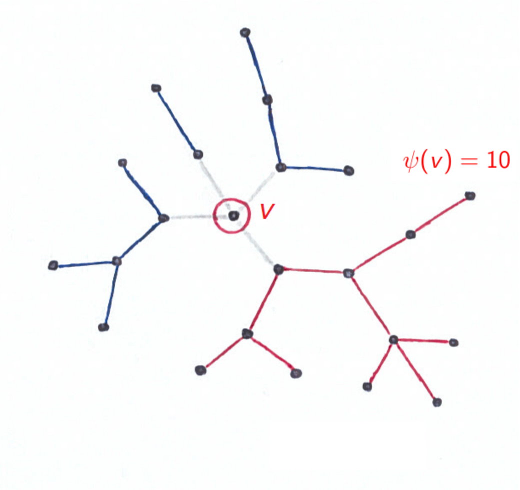
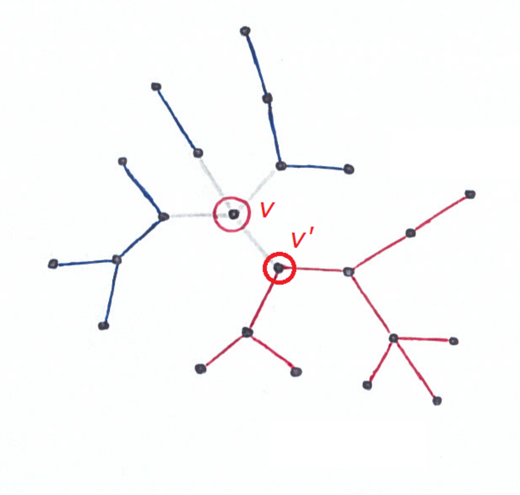


Centrality

Given v, we denote v' to be some vertex in $N(v)$ such that

$$\psi(v) = \left|(T, v)_{v'\downarrow}\right| .$$



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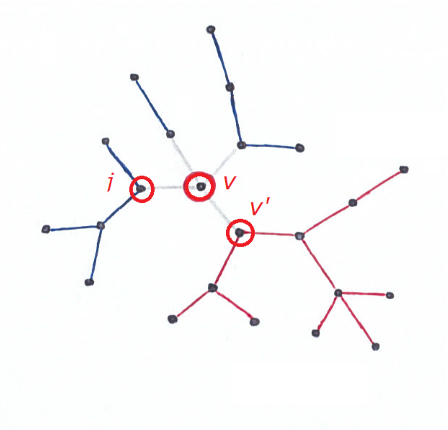

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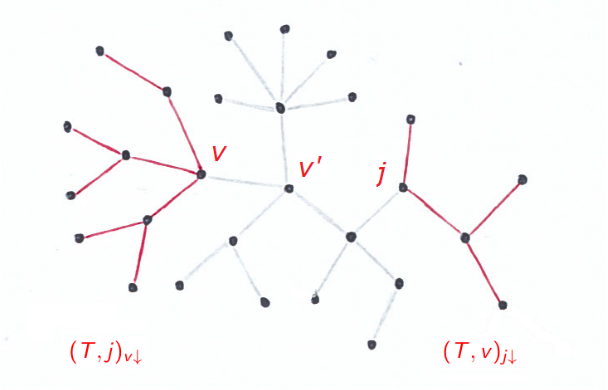
- $|(T,j)_{v\downarrow}| \geq |(T,v)_{j\downarrow}|$;



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More precisely

$$\mathbb{P}\left\{ \max_{\ell\gamma/2 \le j \le \ell(1-\gamma/2)} \psi(j) < \min_{\ell < i \le n} \psi(i) \right\} \ge 1 - \epsilon \ .$$

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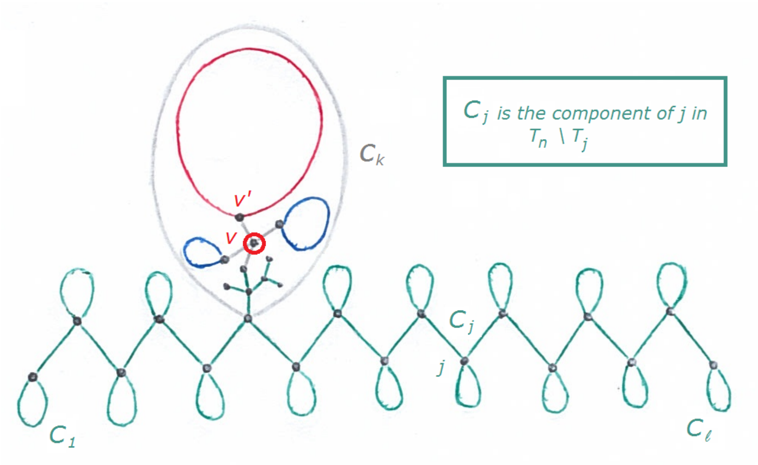
Let us prove that the complement has small probability.

Sketch of the proof

By Union Bound we have

$$\begin{aligned}\mathbb{P}\left\{\min_{\ell<i\leq n}\psi(i)\leq\max_{\ell\gamma/2\leq j\leq\ell(1-\gamma/2)}\psi(j)\right\} &\leq \sum_{j=\gamma\ell/2}^{(1-\gamma/2)\ell}\mathbb{P}\left\{\min_{\ell<i\leq n}\psi(i)\leq\psi(j)\right\}\\ &\leq \sum_{j=\gamma\ell/2}^{(1-\gamma/2)\ell}\sum_{k=1}^{\ell}\mathbb{P}\Big\{\exists v\in C_k\setminus\{k\}:\psi(v)\leq\psi(j)\Big\}\,.\end{aligned}$$

First case: (v', v, j)



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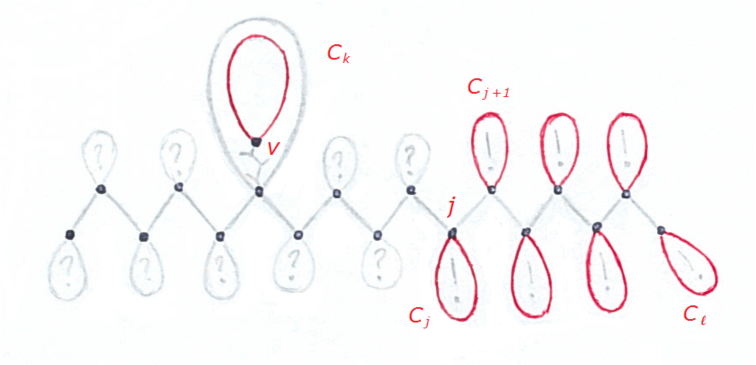

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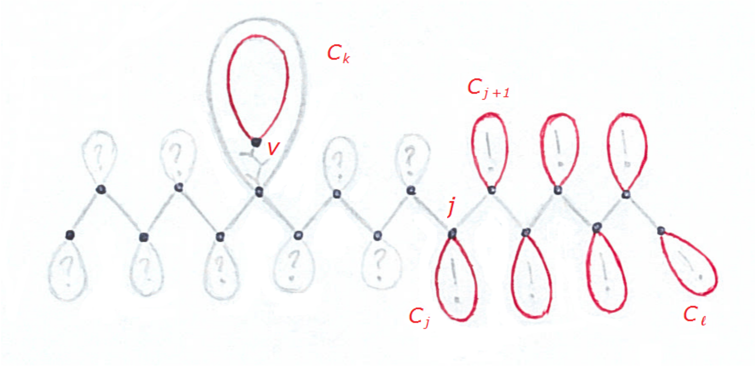

The value of ℓ arise from a optimization of the bounds.

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Thank you